Indian J. Phys. 43, 399—408 (1969)

Temperature distribution in generalized plane Couette flow between two parallel flat plates

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(*Received* 11 *September* 1968)

In this paper expressions for the temperature distributions in a channel bounded by two parallel flat plates (generalized plane Couette flow) are derived when viscous incompressible fluid is flowing through it. The term for dissipation due to friction is not neglected and the rate of heat generation per unit volume (i) varies linearly with time, and (ii) decreases exponentially with time. It is seen in the first case that the temperature below the axis of the channel is more than the temperature above the axis of the channel.

Introduction

The steady flow of a viscous incompressible fluid between two parallel flat plates, one at rest and the other in uniform motion, under constant pressure gradient, is quite well known as generalized plane Couette flow. Pai (1956) has given the velocity and temperature distributions for this flow. However, he has given the soulution of energy equation without considering the rate of heat generation per unit volume in the fluid (other than viscous dissipation). Bhatnagar & Tikekar (1966) have obtained the temperature distribution in a channel bounded by two co-axial circular cylinders. They assumed the rate of heat generation per unit volume as a function of time but did not include the effects of viscous dissipation. Purohit (2967) has obtained the temperature distribution in plane Couette flow between parallel flat plates. He derived the expression for the temperature by assuming the rate of heat generation per unit volume as an oscillatory function of time. Using Laplace transform technique he has obtained the solution of the energy equation which comes out in a form which exhibits the contributions of boundary conditions, dissipation due to friction and the rate of heat generation to the temperature distribution. Recently, the author (1967) has obtained the temperature distribution of a viscous incompressible fluid in a circular pipe when the rate of heat generation per unit volume (i) varies linearly with time, and (ii) decreases exponentially with time, and it has been shown that the points near the axis of the cylinder have higher temperature than those of the points which are far from the axis of the cylinder.

The present paper consists of two parts. In part A the temperature distribution in a channel bounded by two parallel flat plates when viscous incompressible fluid is flowing through it (generalized plane Couette flow), with the rate of heat generation per unit volume varying linearly with time, is discussed. An expression for the temperature distribution is obtained in dimensionless form. This consists of two parts, the one varies linearly with dimensionless time Fourier modulus $T_1 = k't/y_0^2$ and the other is transient part of temperature, which vanishes in the limit as t tends to infinity. It is also seen that the contribution of the transient part is insignificant when $T_1 > 1$.

In part B the temperature distribution in the same channel is studied when viscous incompressible fluid is flowing through it with the rate of heat generation per unit volume decreasing exponentially with time. An expression for the temperature has been obtained taking

$$\frac{1}{\rho c_v} \frac{\partial \theta}{\partial t} = \sum_{m=1}^{\infty} a_m e^{-mt};$$

The result obtained is in complete agreement with similar results obtained by Ballabh (1959) and Sneddon (1951) where Ballabh has obtained the expression for the velocity by using the method of superposability and Sneddon has discussed the heat flow under exponentially decreasing temperature gradient.

Here the expressions for the temperature distributions in both the parts are derived with the conditions that the plates situated at $y = \pm y_0$ (i) have zero initial temperatures, and (ii) are always being kept at zero temperatures.

1. Flow Distribution and Energy Equation

For the case of two-dimensional flow of a viscous incompressible fluid with constant properties, the system of equations for the velocity distribution in steady flow along a xy-plane is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \qquad \text{...(1.1)}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \qquad \text{...(1.2)}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \qquad \text{...(1.3)}$$

where ρ is the density of the fluid and ν is the coefficient of kinematic viscosity.

Now let us consider the flow between two parallel flat plates at a distance $2y_0$ apart, of which one is at rest and the other is moving with constant velocity U. For this flow we have

$$u = u(y),\ v = 0,\ p = p(x).$$

Thus the equation (1.3) becomes an identity and the equations (1.1) and (1.2) assume the forms:

$$\frac{\partial u}{\partial x} = 0, \qquad \text{...(1.4)}$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{...(1.5)}$$

The solution of equation (1.5) under the boundary conditions

$$u = 0 \text{ when } y = -y_0\,; \quad \text{and } u = U \text{ when } y = y_0$$

is

$$u = \frac{U}{2}\left(1 + \frac{y}{y_0}\right) + u_m\left(1 - \frac{y^2}{y_0^2}\right), \qquad \text{...(1.6)}$$

$$\text{where } u_m = -\frac{y_0^2}{2\mu}\frac{dp}{dx}.$$

The energy equation is

$$\rho c_v\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right] = \frac{\partial \theta}{\partial t} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\phi, \qquad \text{...(1.7)}$$

where $\phi = (\partial u/\partial y)^2$ is the energy dissipation function; $\partial\theta/\partial t$ is the rate of heat generation per unit volume in the fluid; c_v and k are the specific heat and the coefficient of heat conductivity of the fluid respectively.

The velocity distribution is steady while the temperature is unsteady The temperature distribution does not influence the flow field of an incompressible fluid with constant properties. We have assumed a fluid having these properties.

PART A

Rate of heat generation per unit volume varies linearly with time

If we assume that temperature is independent of its axial position, then $\dfrac{\partial T}{\partial x} = 0$, and hence equation (1.7) reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_v}\frac{\partial \theta}{\partial t} + k'\frac{\partial^2 T}{\partial y^2} + A - By + Cy^2, \qquad \text{...(2.1)}$$

where

$$A = \frac{\mu U^2}{4\rho c_v y_0^2},$$

$$A = \frac{2\mu U u_m}{\rho c_v y_0^3}$$

$$C = \frac{4\mu u_m^2}{\rho c_v y_0^4}.$$

The term $(A - By + Cy^2)$ in the equation (2.1) is dissipation due to friction and is not neglected in the present investigation.
We now assume that

$$\frac{1}{\rho c_v} \frac{\partial Q}{\partial t} = at. \qquad \ldots(2.2)$$

Equation (2.1) then becomes

$$\frac{\partial T}{\partial t} = at + k' \frac{\partial^2 T}{\partial y^2} + A - By + Cy^2. \qquad \ldots(2.3)$$

Now let $\bar{T} = \int_0^\infty e^{-st}\, T\, dt$ be the Laplace transform of T and let T_0 be the initial value of T.

Multiplying equation (2.3) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{T}}{\partial y^2} - p^2 \bar{T} = -\frac{1}{k'}\left[T_0 + \frac{A}{s} - \frac{By}{s} + \frac{Cy^2}{s} + \frac{a}{s^2}\right], \qquad (2.4)$$

where $\qquad p^2 = s/k'$.

We shall now find T_0.

Initially the rate of heat generation is zero and the temperature is steady in the channel.

Hence $\dfrac{\partial T_0}{\partial t} = 0$ and we obtain $\dfrac{d^2 T_0}{dy^2} = -\dfrac{1}{k'}\left(A - By + Cy^2\right)$ $\qquad \ldots(2.5)$

The solution of equation (2.5) under the boundary conditions

$$T_0 = 0 \text{ when } y = -y_0\,;\; T_0 = 0 \text{ when } y = y_0$$

is

$$T_0 = \frac{A}{2k'}\left(y_0^2 - y^2\right) - \frac{B}{6k'}\left(y_0^2 y - y^3\right) + \frac{C}{12k'}\left(y_0^4 - y^4\right).$$

Substituting this value T_0 in (2.4) we get

$$\frac{\partial^2 \bar{T}}{\partial y^2} p^2 \bar{T} = -\frac{1}{k'}\left[\frac{A}{2k'}\left(y_0^2 - y^2\right) - \frac{B}{6k'}\left(y_0^2 y - y^3\right) + \frac{C}{12k'}\left(y_0^4 - y^4\right) + \frac{A}{s} - \frac{By}{s} + \frac{Cy^2}{s} + \frac{a}{s^2}\right] \quad \ldots(2.6)$$

The boundary conditions for $\bar{T}$ are

$\bar{T} = 0$ when $y = -y_0$; and $\bar{T} = 0$ when $y = y_0$.

The solution of equation (2.6) under the above boundary conditions is

$$\bar{T} = \frac{A}{2k'}\left(\frac{y_0^2 - y^2}{s}\right) - \frac{B}{6k'}\left(\frac{y_0^2 y - y^3}{s}\right) + \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s}\right) + \frac{a}{s^3}\left[1 - \frac{\cosh py}{\cosh py_0}\right] \quad \ldots(2.7)$$

Now applying Laplace inversion theorem, we obtain

$$T = \frac{A}{2k'}\left(y_0^2 - y^2\right) - \frac{B}{6k'}\left(y_0^2 y - y^3\right) + \frac{C}{12k'}\left(y_0^4 - y^4\right) + \frac{1}{2k'}\left(y_0^2 - y^2\right) at - \frac{a}{24k'^2}\left(5y_0^2 - y^2\right)\left(y_0^2 - y^2\right)$$

$$+ \frac{64ay_0^4}{k'^2\pi^5} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} \cdot e^{-\frac{k'(2n+1)^2\pi^2 t}{4y_0}} \cdot \cos\left[\frac{(2n+1)\pi y}{2y_0}\right] \quad \ldots(2.8)$$

At time $t = 0$, $T = \frac{A}{2k'}\left(y_0^2 - y^2\right) - \frac{B}{6k'}\left(y_0^2 y - y^3\right) + \frac{C}{12k'}\left(y_0^4 - y^4\right)$.

Hence from equation (2.8) by putting $t = 0$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} \cos\left[\frac{(2n+1)\pi y}{2y_0}\right] = \frac{\pi^5}{1536}\left(5 - \frac{y^2}{y_0^2}\right)\left(1 - \frac{y^2}{y_0^2}\right).$$

Writing $\frac{y}{y_0} = r$ so that $|r|$ less than 1, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} \cos\left[\frac{(2n+1)\pi}{2} r\right] = \frac{\pi^5}{1536}\left(5 - r^2\right)\left(1 - r^2\right).$$

Putting $r = 0$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

Now we make equation (2.8) dimensionless by introducing

$$\tau = \frac{T}{\theta}, \ \frac{y}{y_0} = r, \ T_1 = \frac{k't}{y_0^2},$$

where θ is a characteristic temperature.

We then get

$$\tau = b_1(1-r^2) - b_2 r(1-r^2) + b_0(1-r^4) + bT_1(1-r^2)$$
$$-\frac{b}{12}\left(5-r^2\right)\left(1-r\right)$$
$$+\frac{128b}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\cdot e^{-\frac{(2n+1)^2\pi^2}{4}T_1}\cdot\cos\left[\frac{(2n+1)\pi}{2}r\right], \qquad \ldots(2.9)$$

where $b_1 = \frac{Ay_0^2}{2k'\theta}$, $b_2 = \frac{By_0^3}{6k'\theta}$, $b_0 = \frac{Cy_0^4}{12k'\theta}$ and $b = \frac{ay_0^4}{2k'^2\theta}$ are clearly dimensionless numbers.

We now take $\tau = \tau_1 + \tau_2$, where

$$\tau_1 = b_1(1-r^2) - b_2 r(1-r^2) + b_0(1-r^4) + bT_1(1-r^2)$$
$$-\frac{b}{12}(5-r^2)(1-r^2),$$

and $\tau_2 = \frac{128b}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\cdot e^{-\frac{(2n+1)^2\pi^2}{4}T_1}\cdot\cos\left[\frac{(2n+1)\pi}{2}r\right].$

The values of τ for different values of r and T_1 have been tabulated.

TABLE 1. $b_1 = 2$, $b_2 = 2$, $b_0 = 2$, $b = 1$

r \ T_1	0.01	0.1	0.2	0.3	0.4	0.8	1.0
−0.9	3.3098	3.3160	3.3219	3.3324	3.3445	3.4052	3.4394
−0.6	3.7888	3.8067	3.8215	3.8533	3.8915	4.0901	4.2036
−0.3	4.3518	4.3736	4.3901	4.4323	4.4842	4.7612	4.9212
0.0	4.0019	4.0256	4.0422	4.0874	4.1435	4.4459	4.6213
0.3	3.2598	3.2816	3.2981	3.3403	3.3922	3.6692	3.8292
0.6	2.2528	2.2707	2.2855	2.3173	2.3645	2.5541	2.6676
0.9	0.7258	0.7320	0.7379	0.7484	0.7605	0.8212	0.8554

TABLE 2. $b_1 = 1, b_2 = 2, b_0 = 1, b = 1$

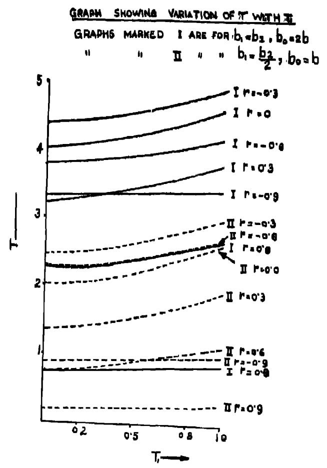
r \ T_1	0.01	0.1	0.2	0.3	0.4	0.8	1.0
−0.9	0.8759	0.8821	0.8880	0.8985	0.9106	0.9713	1.0055
−0.6	2.2784	2.2963	2.3111	2.3429	2.3811	2.5797	2.6932
−0.3	2.4499	2.4717	2.4882	2.5304	2.5823	2.8593	3.0193
0.0	2.0019	2.0256	2.0422	2.0874	2.1435	2.4459	2.6213
0.3	1.3579	1.3797	1.3962	1.4384	1.4903	1.7673	1.9273
0.6	0.7424	0.7603	0.7751	0.8069	0.8451	1.0437	1.1572
0.9	0.1919	0.1981	0.2040	0.2145	0.2266	0.2873	0.3215

TABLE 3. $b_1 = 2, b_2 = 1, b_0 = 1, b=2$

r \ T_1	0.01	0.1	0.2	0.3	0.4	0.8	1.0
−0.9	0.8949	0.9073	0.9191	0.9401	0.9643	1.0857	1.1541
−0.6	2.5344	2.5702	2.5998	2.6634	2.7398	3.1370	3.3640
−0.3	3.0889	3.1325	3.1655	3.2499	3.3537	3.9077	4.2277
0.0	3.0038	3.0512	3.0844	3.1748	3.2870	3.8918	4.2426
0.3	2.5429	2.5865	2.6195	2.7039	2.8077	3.3617	3.6817
0.6	1.7664	1.8022	1.8318	1.8954	1.9718	2.3690	2.5960
0.9	0.5529	0.5653	0.5771	0.5981	0.6223	0.7437	0.8121

The graphs for fixed r ($r = -0.9, -0.6, -0.3, 0.0, 0.3, 0.6, 0.9$) showing the variation of τ with the parameter T_1 have been drawn in two cases $b_1 = 2,\ b_2 = 2,\ b_0 = 2,\ b = 1$; $b_1 = 1,\ b_2 = 2,\ b_0 = 1,\ b = 1$ in the range $T_1 = 0$ to $T_1 = 1$.

The graphs beyond $T_1 = 1$ have not been drawn because τ_2 is very small compared to τ_1 when $T_1 > 1$, hence the transient part is insignificant and τ varies linearly with T_1 in this range. From the graphs and tables of values it is observed that τ increases with T_1 for fixed r. It is also seen that for any value of T_1, the temperature of any point below the axis of the channel is more than the temperature of the point symmetrical to this above the axis of the channel and thus the temperature below the axis is more than the temperature above the axis. The present case is entirely different from that of plane Couette flow (Dube 1968a) and plane Poiseuille flow



(Dube 1968b) because in the last two flows it has been shown that the temperature above and below the axis of the channel is the same.

PART B

Rate of heat generation per unit volume decreases exponentially with time

We assume that

$$\frac{1}{\rho c_v} \frac{\partial \theta}{\partial t} = \sum_{m=1}^{\infty} a_m e^{-mt}. \quad \text{...(3.1)}$$

Equation (2.1) then becomes

$$\frac{\partial T}{\partial t} = \sum_{m=1}^{\infty} a_m e^{-mt} + k' \frac{\partial^2 T}{\partial y^2} + A - By + Cy^2. \quad \text{...(3.2)}$$

Let $\bar{T} = \int_0^{\infty} e^{-st}\, T dt$ be the Laplace transform of T and let T_0 be the initial value of T.

Multiplying equation (3.2) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{T}}{\partial y^2} - p^2 \bar{T} = -\frac{1}{k'}\left[T_0 + \frac{A}{s} - \frac{By}{s} + \frac{Cy^2}{s} + \sum_{m=1}^{\infty} \frac{a_m}{(s+m)} \right],$$

where $p^2 = \dfrac{s}{k'}$.

Here $T_0 = \dfrac{A}{2k'}\left(y_0^2 - y^2 \right) - \dfrac{B}{6k'}\left(y_0^2 y - y^3 \right) + \dfrac{C}{12k'}\left(y_0^4 - y^4 \right)$

as obtained in part A.

The solution of equation (3.3) under the previous boundary conditions is

$$\bar{T} = \frac{A}{2k'}\left(\frac{y_0^2 - y^2}{s} \right) - \frac{B}{6k'}\left(\frac{y_0^2 y - y^3}{s} \right)$$

$$+ \frac{C}{12k'}\left(\frac{y_0^4 - y^4}{s} \right) + \left(1 - \frac{\cosh py}{\cosh py_0} \right) \sum_{m=1}^{\infty} \frac{a_m}{s(s+m)} .$$

Now applying Laplace inversion theorem, we get

$$T = \frac{A}{2k'}\left(y_0^2 - y^2 \right) - \frac{B}{6k'}\left(y_0^2 y - y^3 \right) + \frac{C}{12k'}\left(y_0^4 - y^4 \right)$$

$$- \sum_{m=1}^{\infty} \frac{a_m}{m}\left[1 - \frac{\cos\{(m/k')^{1/2} y\}}{\cos\{(m/k')^{1/2} y_0\}} \right] e^{-mt}$$

$$+ \frac{4}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a_m}{(2n+1)\left[m - \dfrac{k'(2n+1)^2 \pi^2}{4y_0^2} \right]} . e^{-\frac{k'(2n+1)^2 \pi^2 t}{4y_0^2}}$$

$$\times \cos\left[\frac{(2n+1)\pi y}{2y_0} \right]. \qquad \ldots(3.4)$$

The expression (3.4) for the temperature is in complete agreement with similar results obtained by Ballabh (1959) and Sneddon (1951). Ballabh in his paper has obtained the expression for the velocity by using the method of superposability and Sneddon has discussed the heat flow under exponentially decreasing temperature gradient.

The author wishes to express his thanks to Dr. P. L. Bhatnagar for his valuable guidance in the preparation of this paper.

References

Pai, S., 1956 *Viscous Flow Theory, I-Laminar Flow*, D. Van Nostrand Co., Inc., New York.

Bhatnagar, P. L. & Tikekar, V. G. 1966 *Proc. Camb. Phil. Soc.*, **62**, 301.

Purohit, G N. 1967, *Proc. Nat. Inst. Sci.* India, **33A**, 3 & 4, 142.

Dube, S. N. 1967 *Indian J. Phys.*, **41**, 433.

Ballabh, R. 1959 *Proc. of V Congress on Theo. and Appl. Mech.* India.

Sneddon, I. N. 1951 *Fourier Transforms*, Mc Graw-Hill Book Co., Inc., New York.

Dube, S. N. 1968a To appear in *Journal of Technology*, India.

1968b To appear in *Proc. Nat. Inst. Sci.* India.